

INTERACTION CURVES OF LINEARLY INTENSIFYING POLYMERIC MATERIALS UNDER TENSILE-TORSIONAL STRESS

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Abstract Theoretical and experimental research has been performed on the interaction curves and stress paths of crystalline polymeric materials PE and POM under tensile-torsional stress with a linearly intensifying model and in terms of the yield points undergoing Von Mises criterion.

Keywords Crystalline polymeric material, Interaction curve, Tensile, Torsional, Stress

INTRODUCTION

Many researchers^[1-3] have reported on the ideal elastoplastic model from the experiments on a rod of polymeric material under tensile torsional stress (Fig. 1). The basic postulate is that yield stress Y is a constant which is included in the equation $\sigma^2 + 3 \tau^2 = Y^2$. However, experiment showed that some polymeric materials' (POM, PE and PEEK) strain is mainly because of strain hardening and other factors, and that Y is no longer a constant but varies with the kind of material, being in the states shown in Fig. 2. Accordingly, it is obviously not precise if Y is still thought of as a constant in analysis with an ideal elastoplastic model. This paper deals theoretically

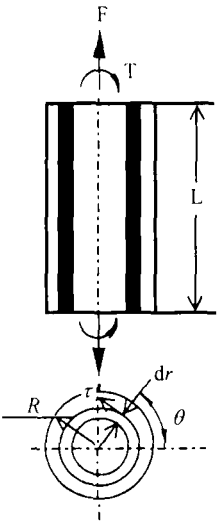


Fig. 1 Model of the twist and the load combination

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and experimentally with the interaction curves and stress paths of crystalline polymeric materials POM and PE under tensile-torsional stress in terms of the yield points undergoing Von Mises criterion and with a linearly intensifying model.

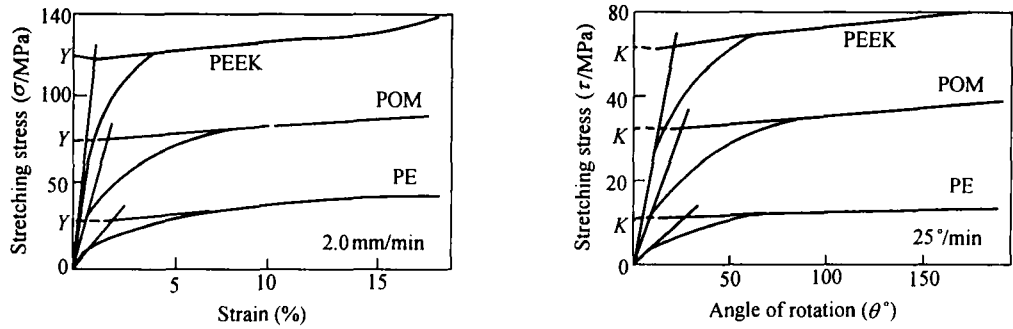


Fig. 2 Experimental curves of pure stretch and pure twist
 $K = 3Y$

LOAD OF PURE TENSILE AND PURE TORQUE

Linearly Intensifying Model

For the convenience of derivation, the linearly intensifying model as shown in Fig. 3a is transformed into one as shown in Fig. 3b, i.e. $Y' = Y + B(\varepsilon - \varepsilon_c)$ into $Y' = Y + a(r - c)$. Where Y' is the yield strength (MPa) at any point of the plasticization stage; Y is the yield strength (MPa) at the critical point ($r = c$) of elastoplasticization; a is a proportional constant (MPa/m); B is a proportional constant (MPa/m); c is the critical radius (mm) of elastoplasticization; r is the radius (mm); ε is the tensile strain and ε_c is the yield strain.

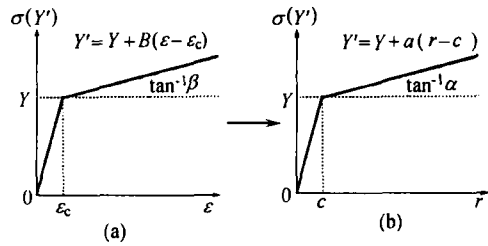


Fig. 3 Linearly intensifying model and their conversion

Stress

Suppose plasticization tends towards the center ($r = 0$) of the sample section from the outermost point ($r = R$). Thus the stress in the elasticity domain ($0 \leq r \leq c$) is as follows:

$$\sigma = E \varepsilon \tag{1a}$$

or

$$\tau = Gr \frac{\theta}{l} \tag{1b}$$

And the stress in the plasticity domain ($0 \leq r \leq R$) is expressed as:

$$\sigma_Y = Y + \alpha (r - c) \tag{2a}$$

or

$$\tau_Y = [Y + a(r - c)]/\sqrt{3} \tag{2b}$$

Pull F and Torque T in Separate Load

$$\begin{aligned}
 F &= \int_0^c E\varepsilon 2\pi dr + \int_0^R [Y + a(r-c)] 2\pi r dr \\
 &= \pi E\varepsilon_c c^2 + \pi(Y-ac)(R^2 - c^2) + \frac{2\pi a}{3}(R^3 - c^3)
 \end{aligned} \quad (3a)$$

$$\begin{aligned}
 T &= \int_0^c G\left(\frac{\theta}{l}\right) r 2\pi r^2 dr + \int_0^R \frac{2\pi}{\sqrt{3}} [Y + a(r-c)] r^2 dr \\
 &= \frac{\pi G}{2} \left(\frac{\theta}{l}\right) c^4 + \frac{2}{3\sqrt{3}} \pi Y (R^3 - c^3) + \frac{\pi a}{2\sqrt{3}} (R^4 - c^4) - \frac{2\pi a}{3\sqrt{3}} c (R^3 - c^3)
 \end{aligned} \quad (4a)$$

Let $c = 0$ when the sample is in complete plasticity, then the above two equations become:

$$F_0 = F|_{c=0} = \pi R^2 Y + \frac{2\pi}{3} a R^3 = \pi Y R^2 \left(\frac{3+\alpha}{3} \right) \quad (3b)$$

$$T_0 = T|_{c=0} = \frac{2\pi}{3\sqrt{3}} Y R^3 + \frac{\pi a}{2\sqrt{3}} R^4 = \frac{2\pi Y R^3}{3\sqrt{3}} \left(\frac{4+3\alpha}{4} \right) \quad (4b)$$

In Eqs. (3) and (4), E and G are the material's elasticity moduli (MPa) of stretch and shear, respectively, R is the radius (mm) of the rod sample, and α is a constant without physical unit, and at $\alpha = a R/Y$ and $a = 0$, $\alpha = 0$, then the above equations are just the same as in the previous Ref. [3].

PULL F AND TORQUE T IN TENSILE-TORSIONAL LOAD

First of all, it is assumed that the linearly intensifying polymeric material still satisfies the Von Mises yield condition:

$$\sigma_Y^2 + 3\tau_Y^2 = (Y')^2$$

The Resultant Stress

In accordance with the method previously illustrated^[3], the resultant stress can be obtained by the linearly intensifying model.

The stress in the elasticity domain ($0 \leq r \leq c$) should be:

$$\sigma = E\varepsilon \quad (5a)$$

$$\tau = G \frac{\theta}{l} r \quad (5b)$$

And the stress in the plasticity domain ($0 \leq r \leq R$) should be:

$$\sigma_Y = \frac{\beta[Y + a(r-c)]}{(\beta^2 + 3r^2)^{1/2}} \quad (6a)$$

$$\sigma_Y = \frac{r[Y + a(r-c)]}{(\beta^2 + 3r^2)^{1/2}} \quad (6b)$$

Where ε is the elastic strain, θ/l is the angular rotation for unit length ($^\circ/\text{mm}$),

$$\beta = 3 \frac{dl}{d\theta} = 3 \frac{dl}{d\theta} \frac{dt}{dt} = 3 \frac{\dot{l}}{\dot{\theta}} \quad (7)$$

$\dot{l}$ is the tensile speed (mm/min), and $\dot{\theta}$ is the twist angular speed ($^\circ/\text{mm}$).

Pull F and Torque T

$$\begin{aligned}
 F &= \int_0^c \sigma 2\pi r dr + \int_0^R \sigma_Y 2\pi r dr \\
 &= \int_0^c E\varepsilon 2\pi r dr + \int_0^R \frac{\beta[Y + a(r-c)]}{(\beta^2 + 3r^2)^{1/2}} \cdot 2\pi r dr
 \end{aligned}$$

$$\begin{aligned}
&= \pi E \varepsilon_c c^2 + \frac{2}{3} \pi \beta (Y - ac) \left[(3R^2 + \beta^2)^{1/2} - (3c^2 + \beta^2)^{1/2} \right] \\
&+ \frac{\pi \beta a}{3} \left[R(\beta^2 + 3R^2)^{1/2} - c(\beta^2 + 3c^2)^{1/2} \right] + \frac{\pi \beta^3 a}{3\sqrt{3}} \ln \left[\frac{\sqrt{3}c + (\beta^2 + 3c^2)^{1/2}}{\sqrt{3}R + (\beta^2 + 3R^2)^{1/2}} \right] \quad (8a)
\end{aligned}$$

$$\begin{aligned}
T &= \int_0^c \tau_2 2\pi r \cdot r dr + \int_c^R \tau_Y 2\pi r \cdot r dr \\
&= 2\pi G \int_0^c \frac{\theta}{l} r^3 dr + 2\pi \int_c^R [Y + a(r - c)] \frac{r^3}{(\beta^2 + 3r^2)^{1/2}} dr \\
&= \frac{\pi G}{2} \left(\frac{\theta}{l} \right)_c c^4 + \frac{2\pi}{27} (Y - ac) \left[(3R^2 - 2\beta^2)(3R^2 + \beta^2)^{1/2} - (3c^2 - 2\beta^2)(3c^2 + \beta^2)^{1/2} \right] + \\
&\quad \frac{\pi}{12} a \left[R(2R^2 - \beta^2)(3R^2 + \beta^2)^{1/2} - c(2c^2 - \beta^2)(3c^2 + \beta^2)^{1/2} \right] + \\
&\quad \frac{\pi a \beta^4}{12\sqrt{3}} \ln \left[\frac{\sqrt{3}R + (3R^2 + \beta^2)^{1/2}}{\sqrt{3}c + (3c^2 + \beta^2)^{1/2}} \right] \quad (9a)
\end{aligned}$$

Let $c = 0$, when the sample section fully yields, then the above equations will be transformed into:

$$\begin{aligned}
F_Y &= \frac{2}{3} \pi \beta \left[(3R^2 + \beta^2)^{1/2} - \beta \right] + \\
&\quad \frac{2\pi \beta a}{3} \left[\frac{R}{2} (\beta^2 + 3R^2)^{1/2} \right] + \frac{\pi \beta^3 a}{3\sqrt{3}} \ln \left[\frac{\beta}{\sqrt{3}R + (\beta^2 + 3R^2)^{1/2}} \right] \quad (8b)
\end{aligned}$$

$$\begin{aligned}
T_Y &= \frac{2\pi}{27} Y \left[(3R^2 - 2\beta^2)(3R^2 + \beta^2)^{1/2} + 2\beta^3 \right] + \\
&\quad \frac{\pi}{12} a R (2R^2 - \beta^2)(3R^2 + \beta^2)^{1/2} + \frac{\pi \beta^4 a}{12\sqrt{3}} \ln \left[\frac{\sqrt{3}R + (3R^2 + \beta^2)^{1/2}}{\beta} \right] \quad (9b)
\end{aligned}$$

According to the principle that the stresses are equal at the critical place, the following equations hold well.

$$E \varepsilon_c = \frac{Y \beta}{(\beta^2 + 3c^2)^{1/2}} \quad (10)$$

$$G \left(\frac{\theta}{l} \right)_c = \frac{c Y}{(\beta^2 + 3c^2)^{1/2}} \quad (11)$$

When $\alpha = 0$, the above equations are similar to those in the previous Ref. [3].

INTERACTION CURVE EQUATIONS

Interaction Curve Equations

When the materials section fully yields, from Eqs. (3b), (4b), (8b) and (9b), we can obtain the following interaction curve equation group:

$$\begin{aligned}
\left(\frac{F_Y}{F_0} \right) &= \frac{2}{3} \left(\frac{3}{3 + \alpha} \right) \left(\frac{\beta}{R^2} \right) \left[(3R^2 + \beta^2)^{1/2} - \beta \right] + \frac{1}{3} \left(\frac{3\alpha}{3 + \alpha} \right) \left(\frac{\beta}{R^2} \right) (3R^2 + \beta^2)^{1/2} + \\
&\quad \frac{1}{3\sqrt{3}} \left(\frac{3\alpha}{3 + \alpha} \right) \left(\frac{\beta}{R} \right)^3 \ln \left[\frac{\beta}{\sqrt{3}R + (3R^2 + \beta^2)^{1/2}} \right] \quad (12) \\
\left(\frac{T_Y}{T_0} \right) &= \frac{1}{3\sqrt{3}} \left(\frac{4}{4 + 3\alpha} \right) \left(\frac{1}{R^3} \right) \left[(3R^2 - 2\beta^2)(3R^2 + \beta^2)^{1/2} + 2\beta^3 \right] +
\end{aligned}$$

$$\frac{\sqrt{3}}{2}\left(\frac{\alpha}{4+3\alpha}\right)\left(\frac{1}{R^3}\right)(2R^2-\beta^2)(3R^2+\beta^2)^{\frac{1}{2}} +$$
$$\frac{1}{2}\left(\frac{\alpha}{4+3\alpha}\right)\left(\frac{\beta}{R}\right)^4 \ln\left[\frac{\sqrt{3}R+(3R^2+\beta^2)^{\frac{1}{2}}}{\beta}\right]$$

(13)

It is known from the above observation that Eqs. (12) and (13) are the functions of α , R and β . If the above two equations are combined, the function relation $(T_Y/Y_0) = f[(F_Y/F_0), \alpha]$ will be obtained through the elimination of R and β . But it is extremely difficult to do so in mathematics. Only when $\alpha = 0$ can it be simplified to

$$\left(\frac{T_Y}{T_0}\right) = \frac{1}{2}\left(1 - \frac{F_Y}{F_0}\right)^{\frac{1}{2}}\left(2 + \frac{F_Y}{F_0}\right)$$
$$\left(\frac{T_Y}{T_0}\right)^2 + \left(\frac{F_Y}{F_0}\right)^2 = 1 - \frac{1}{4}\left(\frac{F_Y}{F_0}\right)^3$$

(14a)

Because of $0 \leq T_Y/Y_0 \leq 1$ and $0 \leq F_Y/F_0 \leq 1$, when $\alpha = 0$ the above equation can be transformed into a circle whose radius is approximately equal to 1:

$$\left(\frac{T_Y}{T_0}\right)^2 + \left(\frac{F_Y}{F_0}\right)^2 \approx 1$$

(14b)

That is an ideal elastoplastic model. Hence it is proved that when $\alpha \neq 0$ the interaction curve composed of a point group $(F_Y/F_0, T_Y/Y_0)$ is irrelevant to either R or β . Figure 4 gives the interaction curves in terms of $R = 3.5$ mm and $R = 7$ mm and different values of β (see Table 1) and values of α . It turns out that the curves of $R =$

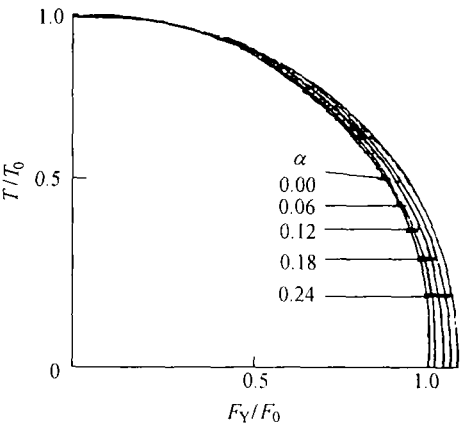


Fig. 4 Relationship between interaction curves and values of α
 $R = 3.5$ mm, 7 mm

Table 1. Experimental conditions for PE and POM under resultant stress			
	β (mm)	Stretch speed (mm/min)	Twist speed (°/min)
A	1.15*	0.167	25.0
B	1.91*	0.167	15.0
C	4.44	0.5000	25.0
D	4.73	0.500	15.
E	10.74	0.500	8.0
F	21.49	1.000	8.0
G	42.98	2.000	8.0

* No condition tests have been carried out for the cases marked with asterisk

3.5 mm and $R = 7$ mm coincide with one another. But the existence of α is the cause of close relation between the interaction curve of the linearly intensifying model and the material characteristic α , which is just the important point or the difference between the linearly intensifying model and the ideal elastoplastic model ($\alpha = 0$).

The Generalized Equation of the Interaction Curve

With a tensile-torsional load and in the case of there are both elasticity and plasticity in the sample section, i.e. elasticity happens in a domain of $0 \leq r \leq c$ and plasticity happens in a domain of $0 \leq r \leq R$, from Eqs. (3b), (4b), (8a) and (9a) another equation group is obtained:

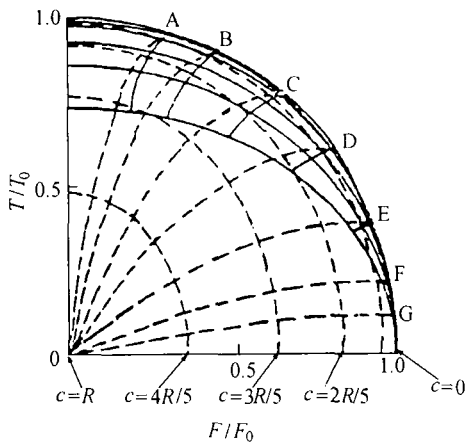
$$\begin{aligned} \left(\frac{F}{F_0}\right) = & \left(\frac{3}{3+\alpha}\right) \frac{\beta}{(3c^2 + \beta^2)^{1/2}} \left(\frac{c}{R}\right)^2 + \\ & \left(\frac{2}{3+\alpha}\right) \left(1 - \frac{c}{R}\alpha\right) \left(\frac{\beta}{R^2}\right) \left[(3R^2 + \beta^2)^{1/2} - (3c^2 + \beta^2)^{1/2}\right] + \\ & \left(\frac{\alpha}{3+\alpha}\right) \left(\frac{\beta}{R^3}\right) \left[R(3R^2 + \beta^2)^{1/2} - c(3c^2 + \beta^2)^{1/2}\right] + \\ & \frac{\alpha}{\sqrt{3}(3+\alpha)} \left(\frac{\beta}{R}\right)^3 \ln \left[\frac{\sqrt{3}c + (3c^2 + \beta^2)^{1/2}}{\sqrt{3}R + (3R^2 + \beta^2)^{1/2}} \right] \end{aligned} \quad (15)$$

$$\begin{aligned} \left(\frac{T}{T_0}\right) = & \left(\frac{3\sqrt{3}}{4+3\alpha}\right) \frac{c^4}{R^3(3c^2 + \beta^2)^{1/2}} + \\ & \left(\frac{4}{3\sqrt{3}(4+3\alpha)}\right) \left(1 - \frac{c}{R}\alpha\right) \left(\frac{1}{R^3}\right) \cdot \\ & \left[(3R^2 - 2\beta^2)(3R^2 + \beta^2)^{1/2} - (3c^2 - 2\beta^2)(3c^2 + \beta^2)^{1/2}\right] + \\ & \frac{\sqrt{3}\alpha}{2(4+3\alpha)} \left(\frac{1}{R^4}\right) \left[R(2R^2 - \beta^2)(3R^2 + \beta^2)^{1/2} - c(2c^2 - \beta^2)(3c^2 + \beta^2)^{1/2}\right] + \\ & \frac{\alpha}{2(4+3\alpha)} \left(\frac{\beta}{R}\right)^4 \ln \left[\frac{\sqrt{3}R + (3R^2 + \beta^2)^{1/2}}{\sqrt{3}c + (3c^2 + \beta^2)^{1/2}} \right] \end{aligned} \quad (16)$$

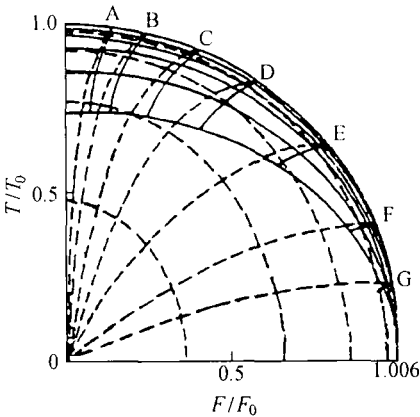
The following can be inferred from Eqs. (15) and (16). (1) When $\alpha = 0$, this group accords with the ideal elastoplastic model in the previous Ref. [3]; (2) When $\alpha \neq 0$, the above equations are related to α , R , β and c , respectively. We, therefore, call the curve of point group $(F/F_0, T/T_0)$ with given α , R , a certain c , and varying β the generalized interaction curve. Now the group of (15) and (16) is called the generalized interaction curve equations.

Here is an example. When $\alpha = 0.04, 0.1$, $R = 3.5$ mm, 7 mm and $c = 0, R/5, 2R/5, 3R/5, 4R/5$ and R , the curve of the point group $(F/F_0, T/T_0)$ varying with β (see Table 1) is a cluster of 1/4 elliptical slim real lines (see Fig. 5). These lines are called the generalized interaction curves. If the first term (the effect of the elastic term) in Eqs. (15) and (16) is dropped out, then a group of 1/4 elliptical slim dotted lines shown in Fig. 5 are obtained, which are called the generalized plastic interaction curves.

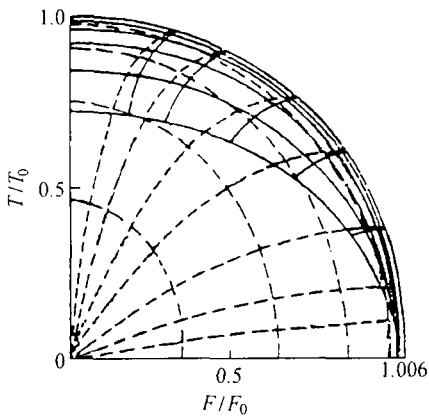
From Fig. 5 the following points are known. (1) Neither the generalized interaction curves (the 1/4 elliptical slim real lines), nor the generalized plastic interaction curves (the 1/4 elliptical slim dotted lines) are related to R and β , and the curves of $R = 3.5$ mm are just the same as those of $R = 7$ mm. (2) The generalized interaction curves have a connection with the generalized plastic interaction curves. The smaller the value of α , the closer the generalized interaction curves (the 1/4 elliptical slim real line) will approach the margin. The same is true for the generalized plastic interaction curve (the 1/4 elliptical slim dotted lines).



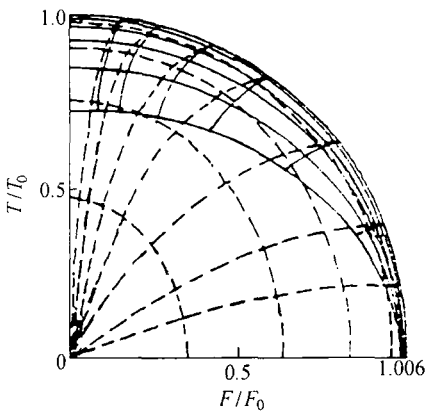
(a) $\alpha = 0, R = 3.5 \text{ mm}$



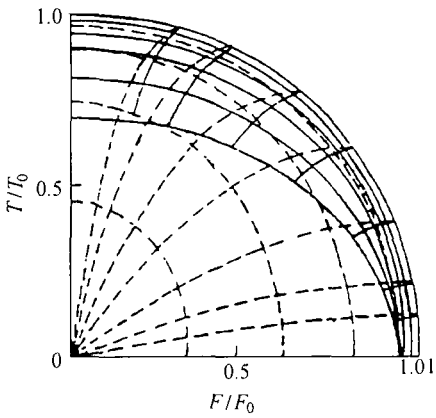
(d) $\alpha = 0, R = 7 \text{ mm}$



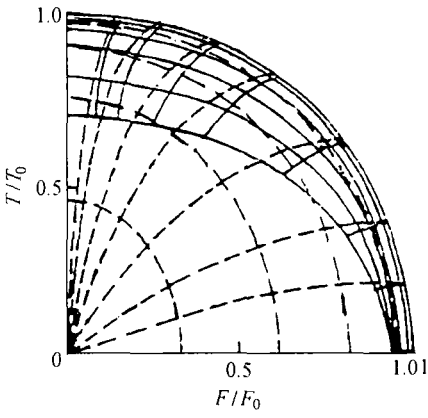
(b) $\alpha = 0.04, R = 3.5 \text{ mm}$



(e) $\alpha = 0.04, R = 7 \text{ mm}$



(c) $\alpha = 0.04, R = 3.5 \text{ mm}$



(f) $\alpha = 0.04, R = 7 \text{ mm}$

Fig. 5 Theoretical curves of different values of α and values of R

The Curve Equation of Stress Path

From Eqs. (15) and (16) it is know that when R, α and β are given, the point $(F/F_0, T/T_0)$ varies with c , to form a

radial curve starting from the origin and tending towards the circumference. And this radial line is called the stress path curve or the complete stress path curve. Eqs. (15) and (16) are now called a stress path curves equation group or a complete stress path equation group.

When $\alpha = 0$, this equation group accords with the ideal elastoplastic model in the previous Ref. [3]. When $\alpha = 0.04, 0.1, R = 3.5 \text{ mm}, 7 \text{ mm}$, and β (see Table 1) is a certain value, the point $(F/F_0, T/T_0)$ varies with $c = 0, R/5, 2R/5, 3R/5, 4R/5, R$ to form a cluster of slim real lines starting from the origin and extending in the direction of the circumference (see Fig. 5), which are the stress path curves. If in Eqs. (15) and (16) the first term (the effect of the elastic term) is omitted, we can get a cluster of slim radial dotted lines (see Fig. 5 starting from the origin, which are called the plastic stress path curves. When α is large, both the two sorts of stress path curves deflect towards the horizontal axis (F/F_0) . But this deflection is much smaller than that attributed to the decrease of R , and even negligible. It is found from Fig.5 that when the values of R are equal, the stress path curve of $\alpha = 0.1$ deflects slightly towards the horizontal axis as compared with that of $\alpha = 0.04$. And when curves of α are equal, the curve of $R = 3.5 \text{ mm}$, compared with that of $R = 7 \text{ mm}$, tends towards the horizontal axis at a high speed. It shows that the slim rod undergoes stretch instead of twist in comparison with each other. When $\beta \rightarrow \infty$, the stress path curve fits the horizontal axis (F/F_0) ; when $\beta \rightarrow 0$ the path curve fits the vertical axis (T/T_0) .

By summarizing the above interaction curve, the generalized interaction curve and the stress path curve, it can be seen that: (1) the interaction curve has much to do with the used model (depending on whether α is zero) but it has nothing to do with either the size (R) of the sample section or the experimental condition (β); (2) the generalized interaction curve (including the generalized plastic interaction curve) is relevant to the used model (depending whether α is zero) and to the elastoplastic radius c_i , but it is irrelevant to either the size (R) of the sample section or the experimental condition (β); (3) the stress path curves including the plastic stress path curves has no connection with the used model (depending on whether it is zero) while the size (R) of the sample section and the experimental condition (β) make a great difference from the stress path curve.

It thus reveals that they are all affected by α , and that the only difference is the degree to which they are affected. But how about the fitting of this theory with the experimental result of the linearly intensifying polymeric material?

EXPERIMENTAL

Experimental materials: two kinds of rods of POM and PE, 15 mm in diameter, cut into the shape shown in Fig. 6. Instrument: UtM-57 Japan universal service platform for resultant stress. Conditions: ambient temperature: $23 \pm 1^\circ\text{C}$, the experimental conditions (C, D, E, F and G) for resultant stress are shown in Table.1. Results: the sign “●” shown in Fig. 7 is the stress path, which means the crosspoint $(F/F_0, T/T_0)$ of the stress ratio (F/F_0) and the twist ratio (T/T_0) that were measured at the same time.

In Fig. 7 the radial dash dot line and the solid radial line indicate the stress path curve of $\alpha = 0$ and that of $\alpha = 0.1$ (neglecting of the effect of the elastic term), respectively.

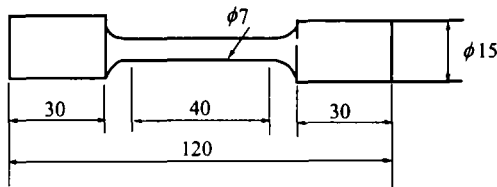


Fig. 6 Shape of sample rod for experiment

ANALYSES AND CONCLUSIONS

The fitting of the theoretical stress path curve can be known from Fig. 7. The smaller the β is, the closer will be the theoretical stress path curves of $\alpha = 0.1$ approaches to the experimental result than that of $\alpha = 0$. And the

difference decreases with the increase of β . It follows that an optimum value of α is certain to be found to make theory fit better with practice.

The fitting with the interaction curve can be seen from Fig. 7. Most of the outermost points of POM and PE experimental stress paths lie between the theoretical curve of $\alpha = 0$ and that of $\alpha = 0.1$. This also indicates that there is certain to be an optimum value of α to fit the experimental result quite well with the theoretical curve of this α . The discussions of other results of experiment and theory are omitted because of the similarities between this paper and the previous Ref. [3].

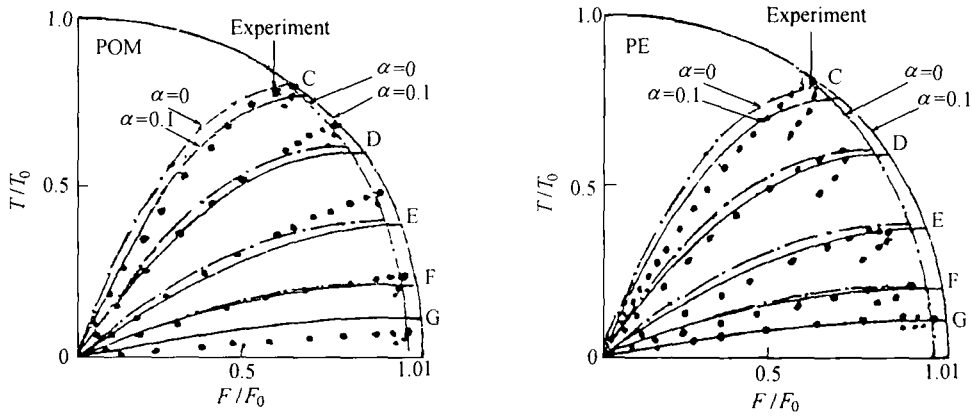


Fig. 7 Comparison between theoretical curves and experimental results

A conclusion can be drawn from the above exposition that the analysis of linearly intensifying polymeric materials such as POM and PE with the linearly intensifying model is more precise than that with ideal elastoplastic model.

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